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Limits and Loci:
On Motion's Role in the Definition of Curved Lines

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INTRODUCTION

It is not usually appreciated that mathematics, as known today, has changed drastically from that of the ancients. Perhaps a child assumes, owing to his perception of the immutability of mathematics, that its method and tools must themselves have existed as long as the science itself. Yet, modern mathematics only exists because of the works of great minds in the seventeenth century. These men, by reshaping the discipline of Euclid, were able to use mathematics to solve problems that were impossible before.

One of these great revolutionary thinkers was Rene Descartes. He created a method that allowed geometricians to consider curved lines that once were unintelligible. These new curves, he claimed, were rejected by the ancients since they required motion to describe them and therefore were mechanical. Yet Descartes craftily omitted from his geometry other curves "described by two separate motions"[1] maintaining that it was only the lack of unity in the motions that made them mechanical.

Throughout this division, it is clear Descartes has no qualms about defining curved lines or any object of mathematics by means of motion. Yet, Saint Thomas Aquinas' view on mathematics was that its objects are abstracted from the material world, and thus without motion. If this is true, then the very foundation of Descartes' system would be on unstable ground.

In my thesis, I intend to prove that Descartes' attempt to found geometry on motion was not only unnecessary, but illegitimate. For curved lines are rightfully objects of

[1] Descartes, Rene. *The Geomerty of Rene Descartes.* Bk. II. Pg. 44.

mathematics because they are defined by mathematical laws, which are unchanging. I will do this, first, by explaining Descartes' argument for defining curved lines by motion, after which I will show how calculus supports this claim. Then, with the help of Saint Thomas, I will make clear why this is a mistake. Next, in order to prove definitively that curved lines are not by nature mechanical, I will show that curves are not distinguished by what is variable but by law. Finally, I will provide an explanation of why Descartes and many other modern mathematicians made this error; namely, they did not see that motion is invoked in the definition of a curved line only to illustrate the line's unity in an apprehensible way for man's mode of knowing.

I. The Generation of the Curved Line

Descartes portrays himself as the first mathematician to consider curved lines beyond the conic sections as objects of geometry. The ancients, at least as characterized by him, rejected these more composed curved lines as mechanical. These new curved lines, and ultimately his Cartesian coordinate system which would help understand them, was the product of his solution to the locus problem. A locus is a line's or surface's position which has the same property at every place. For instance, the problem which Descartes addresses is the three or four line locus problem. In this problem, one is given three or four lines' positions in respect to each other, and then one must find every point which satisfies a proportion of the lines drawn from those given lines to this point.[2] Now, since there can be "an infinite number of

[2] For a more detailed account, see Descartes, Rene. *The Geomerty of Rene Descartes.* Bk. I. Pg. 22.

different points satisfying these requirements,"[3] all of the points must then lie on some point, line, figure, or area. This answer would be the locus.

The place where Descartes surpassed previous mathematicians regarding the locus problem is in his realization that it could be answered through the relation, or equation, of a horizontal axis, known by the independent variable *x*, and a vertical axis, known by the dependent variable *y*. For he says that each point on the curve has a "definite relation to all points of a straight line, and that this relation must be expressed by means of a single equation."[4] This means that at every point on the x-axis there will be a corresponding point on the curve which will be the distance of *y* away from the x-axis. And this distance will be governed by its relation to *x*.[5] This concept has morphed into the modern idea of the function, which at least at first is just an equation that has isolated the dependent variable on one side of the equation.[6] This relation gave mathematicians something definite to study in the locus.

For this reason, he declared that he was astonished that the ancients did not "distinguish, (beyond the conic sections) between different degrees of these among more complex curves, nor [did he] see why they called the latter mechanical, rather than geometrical."[7] He said that they did this because an "instrument has to be used to describe them."[8] In other words, the ancients rejected more composed curved lines since they could only be considered as the motion of a machine, which they believed was not an object of geometry but mechanics.

[3] Descartes, Rene. *The Geomerty of Rene Descartes.* Bk. I. Pg. 22.
[4] Ibid. Bk. II. Pg. 48.
[5] E.g., $y=x^2$ and $x=2$, then $y=4$
[6] E.g., $y=x-1$ is the same equation as $1=x-y$ but has been expressed as a function.
[7] Ibid. Bk. II Pg. 40.
[8] Ibid. Bk. II. Pg. 40

This was, for Descartes, no valid reason to omit these curves from geometry. For he says that if one excludes his more composed lines, one will be forced to exclude the circle and straight line since "these cannot be described on paper without the use of compasses and a ruler."[9] By this, he implies that motion is not as foreign to mathematics as previously thought. Sir Isaac Newton even echoes this sentiment when he says that, because geometry relies on the ability to describe the circle and straight line, "geometry is founded upon mechanical practice, and is nothing other than that part of universal mechanics."[10]Therefore, if one only relinquishes the conviction that motion is beyond the realm of mathematics, then it would be reasonable to allow Descartes' more composed curves into geometry since they only require more composed instruments to describe.

Descartes, however, did not solely confines these lines to being defined by literal instruments. Even when there is no instrument involved, the curve would be a product of an imagined motion. For instance, the motion Descartes prescribed for the generation of curved lines is the imagined motion of two lines, which he remarked: "two or more lines can be moved, one upon the other, determining by their intersection other curves."[11] This motion can be understood as a horizontal growth of a line and a vertical growth of a second line, which is determined by a set proportion to the first line. Nevertheless, it is only with the introduction of the variable that a motion like this can be conceived. Indeed, the notation used for these lines, as said before, is x and y. Hence, the motion can be understood as x expanding from the origin and y expanding vertically in some given proportion to x, this proportion being given by an

[9] Descartes, Rene. *The Geomerty of Rene Descartes.* Bk. II. Pg. 43.
[10] Newton, Isaac. *Philosophiae Naturalis Principia Mathematica*. Pg. 1.
[11] Descartes, Rene. *The Geomerty of Rene Descartes*. Bk. II. Pg. 43.

equation. Thus, it is evident that Descartes has gone beyond understanding curved lines by the motion of instruments to a purely imaginary motion.

Yet, despite his ridiculing of the ancients, he still rejects curves that are "conceived of as described by two separate movements whose relation does not admit of exact determination."[12] He claims that because these motions have no determined relation, and because geometry "is precise and exact,"[13] these curves must be included in mechanics rather than geometry. It is these curves that have in modern times come to be known as transcendental functions since they transcend the bounds of Descartes' algebraic method.

At face value, what Descartes stated seems true because a curve, which is a locus, is the infinity of solutions to some locus problem. Given that Aristotle says, "motion is supposed to belong to the class of things which are continuous; and the infinite presents itself first in the continuous,"[14] one might say the infinity present in curves is the product of the motion that describes them. Euclid, in contrast to this, never relied on infinity and as a result his geometry does not need to invoke motion. Further, transcendental functions have an even greater infinity than algebraic functions, given that they require an infinite number of algebraic operations.[15] Thus, if one thinks that an infinity of points is only knowable by motion, then it is fitting that Descartes says transcendental curved lines are the result of a greater amount of motion than algebraic curved lines.

[12] Descartes, Rene. *The Geomerty of Rene Descartes.* Bk. II Pg. 44.
[13] Ibid. Bk. II. Pg. 43.
[14] Aristotle. Physics. Bk. III. Lines200b17-18
[15] For instance, the transcendental function Sin(x) is equal to the infinite series ($\frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - \cdots$)

Despite this rationale for agreeing with Descartes' barring of transcendental curves from mathematics, modern mathematics has readily accepted them among its ranks. Although the modern mind might already take for granted this inclusion of transcendental functions, one needs to assess whether these curves were justifiably brought in given Descartes's exclusion of them. Yet, the more fundamental question that persists is whether Descartes was right in allowing his curves, if they must be defined by motion, into the science at all. Yet, one might also find that it was unnecessary to reduce curves to the motions of two lines. If so, it follows that Descartes' division of curved lines collapses.

II. Motion in Calculus

Before proceeding to show the error in Descartes' argument, one must first perceive a further motive for agreeing with him; namely calculus, which is conceivably the most pressing reason to define the curve by motion. This branch of mathematics was created since Descartes' methods were impractical, if not ineffective, at finding the tangent at any given point on the curve or the area under the curve. Calculus solves these problems using the limit, which can be thought of as the first boundary which a thing does not move past.[16] Motion clearly seems to be essential to the definition of the limit or at least to one's knowledge of it. For the limit is thought of by the movement by which one derives it, as made clear by its notation.[17] Another

[16] This notion of the limit can be grasped from Newton's first lemma. Newton says "quantities, and also ratios of quantities, which for any finite time continually tend towards equality, and before the end of that time approach nearer to each other than by any given difference, become ultimately equal." The limit in this scenario will be the quantity or ratio of quantity which another quantity is both tending towards and "ultimately" becoming equal to. Thus, the former quantity, which is the limit, acts as a boundary for the latter's change. (Newton, Isaac. Philosophiae Naturalis Principia Mathematica. Bk. 1 Pg. 29.)

[17] E.g., $\lim_{x \to a} f(x) = F$. The limit F is defined by the progression of x to the value a.

sign of this is the reliance of this notation on the use of the variable (similar to Descartes' use of the variable for understanding his mechanisms, as stated above).

For instance, the limit is used in differential calculus to allow one to find the tangent at any point on a given curve. Differential calculus's method for doing this is by taking the limit of the slope of a vanishing secant. This may be imagined by rotating the secant until the line becomes tangent to the curve. The secant can never actually become a tangent; that would go against its definition; yet, this is the reason the tangent is the limit of this motion. The idea of a limit is useful here because one can see that there is some fixed ratio that the difference quotient cannot go past. Likewise, integral calculus is the method of discovering the area under a curve, which is done by taking the limit of a vanishing polygon sum. Once again, the limit is being used to see that as the rectangles shrink into nothingness, the sum approaches some definite value which is the area under the curve. Since the limit is necessary to arrive at both these important properties of curved lines, it is not unreasonable to say that motion itself must be essential to curved lines.

Additionally, calculus has often been thought of as the science of change. This is because calculus proved itself to be incredibly beneficial for understanding the motion of bodies. Since the derivative of the curve expresses the rate of change at any given point, the derivative of a function expressing the relation of distance traversed to time will be the body's velocity, and its second derivative is its acceleration. The former is because the rate of change of the distance in respect to the time is the velocity. Likewise, the rate of change with respect to the time of the velocity is acceleration. Clearly then, curved lines have a direct connection to motion in reality.

Thus, the science of calculus conforms well to Descartes' use of motion for apprehending curved lines. In fact, as shown above, calculus and its notion of the limit has flowed directly out of Descartes' new methods. Hence, it serves as an argument in his favor of defining curved lines by motion. However, it shall now be seen why this is a false understanding of curved lines.

III. The Abstraction of the Mathematicals

With Descartes' division, one is left wondering why the ancients segregated motion to mechanics. In fact, Descartes's division collapses if the ancients were right. Saint Thomas Aquinas advocates for the ancients' view in his treatise *the Division of the Speculative Sciences*. In this work, he justifies why Boethius divides the sciences into natural science, mathematics, and divine science or metaphysics.

To begin, Saint Thomas observes that the sciences differ by what is "essential to the [their] object as object."[18] This is because science is a habit of the mind, and habits must be distinguished by their object. For instance, in sensation "it is incidental to a sense object as such whether it be an animal or plant. Accordingly, the distinction between the senses is not based upon this difference but rather upon the difference between color and sound."[19] Or to clarify, if both the ears and the eyes can perceive an animal or a plant, then the sensation of an animal or plant really does not distinguish the eyes or the ears. For this reason, Aristotle in his *De Anima* classifies animal, plant, or any substance as accidental sensibles; or in other words accidentally

[18] Aquinas, Thomas. *The Division and Methods of the Sciences*. Pg. 13.
[19] Ibid. Pg. 13.

the object of sensation. Even the common sensibles,[20] though not accidentally the objects of the senses, will not distinguish the senses since they are common to them all. Now what is properly the object of sight is color, and for hearing is sound. Therefore, if one wishes to distinguish between the senses, one must look at their proper objects. Similarly, the sciences, as habits by which man is cable of understanding truth, will differ by what is proper to their objects.

With this distinction in mind, Saint Thomas elaborates on how the objects of sciences are the same and how they differ. Now, what is the same about the objects is that they are immaterial and necessary. This is because the intellect is immaterial and "science treats of necessary matters."[21] Thus, the objects of the sciences must be immutable, since "everything changeable is, as such, able to be or not be, either absolutely or in a certain respect"[22] and thus are not necessary. This leads him to say:

> Consequently, separation from matter and motion, or connection with them, essentially belongs to an object of speculation, which is the object of speculative science. As a result, the speculative sciences are differentiated according to their degree of separation from matter and motion.[23]

In other words, what distinguishes the objects of the sciences is the way in which they can be considered without motion and matter. This, however, is not to say that science cannot consider things in motion and matter. This is to say that what is necessary and immutable is not motion or matter but what either exists in them or in degrees of separation from them.

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[20] "'Common sensibles' are movement, rest, number, figure, magnitude; these are not particular to any one sense, but are common to all." (Aristotle, *De Anima.* Bk. II. Line 418a16-19

[21] Aquinas, Thomas. *The Division and Methods of the Sciences*. Pg. 13.

[22] Ibid. Pg. 13.

[23] Ibid. Pg. 14.

There are two ways for this varying degree of separation to occur: either by the object not relying on matter for its being, or by it being intelligible without motion and matter. The former is asserting that some objects of science do not exist in matter; while the latter is proposing that some objects of science do not need the sensible to be understood. By this division, the sciences may be divided into three. The objects that rely on matter for both their understanding and their being are the objects of natural science. Next, mathematics' objects rely on matter for their being, but not for their being understood. Finally, the science that has an object that does not rely on matter for either its understanding or being is Divine science or Metaphysics. And there cannot be a fourth category since "there can be nothing that depends on matter for its being understood but not for its being."[24]

But what does it mean for an object of mathematics not to require motion and matter to be understood, yet still rely on matter for their being? This inquiry is answered in article three of question five, where Saint Thomas gives a more detailed account of what his division of the sciences entails for the science of mathematics. In this account, he proceeds to explain what it means for the intellect to abstract the mathematicals. Saint Thomas commences his investigation into abstraction by stating the two operations Aristotle gives for the intellect, which are "understanding of the indivisibles,' by which it knows what a thing is, and another by which it joins and divides… by forming affirmative and negative statements."[25] The first is how one understands universals such as man or animal. The second operation concerns the being of a thing. However, true abstraction cannot occur from this second operation unless what is

[24] Aquinas, Thomas. *The Division and Methods of the Sciences*. Pg. 14.
[25] Ibid. Pg. 34-35.

abstracted is already separate in reality. Consequently, this form of abstraction is called separation and is the way the objects of metaphysics are derived.

The abstraction of things united in reality must then be the result of the first operation of the intellect. Abstraction cannot take place, though, if what is abstracted is incapable of being understood without what it is abstracted from. For instance, "when the nature itself is related to, and depends on something else, with regard to that which forms the definition (ratio) of the nature, and through which the nature itself is understood, clearly we cannot know the nature without that other thing."[26] An example of this is how "it is necessary to include flesh and bone in the definition of man."[27] However, when a "thing does not depend on another with regard to that which forms the definition of the nature, then the intellect can abstract the one from the other so as to know it without the other."[28] Saint Thomas says this is not only said of stone and man, though these are not properly said to be abstracted since they are not united in reality, but also "whether they are joined as part and whole… or even if they are joined as form to matter and accident to subject."[29]

The abstraction of the objects of mathematics must be done by abstracting form from matter. For, mathematics principally concerns quantity which is what Saint Thomas says is the first accident to inhere in substance.[30] Because of this, "quantity can be thought of in substance before the sensible qualities"[31] and thus can be abstracted from them. For instance, it is

[26] Aquinas, Thomas. The Division and Methods of the Sciences. Pg. 36.
[27] Ibid. Pg. 14.
[28] Ibid. Pg. 36
[29] Ibid. Pg. 36-37
[30] Ibid. Pg. 37-38
[31] Ibid. Pg. 38.

impossible for a person to draw any straight line or circle perfectly. Yet, one may still follow and understand a proposition demonstrated using a diagram. This is possible because mathematics does not study the sensible nor do mathematicals include sensible matter in their definition.

This, however, does not mean the mathematicals, being by their natures abstracted from matter, only exist in the intellect. One still must affirm that the science of mathematics in the mind corresponds to reality given that man has abstracted it from reality. This is what Saint Thomas means when he says, "there is no disagreement between his intellect and reality, because even in reality what belongs to the nature of a line does not depend upon that which makes matter sensible, but vice versa."[32] Expressly, the quotation here is stating that not only do the mathematicals exist in sensible objects, but the sensible depends on the mathematical. For, color and other qualities must exist on an object of some dimensions. It would be impossible for an object to lack quantity (to be for instance a point) and then be sensible. Additionally, this clarifies why Saint Thomas claimed that quantity is the first accident. Thus, the mathematicals really do exist in matter.

Therefore, when Saint Thomas says the mathematicals do not require motion and matter to be understood, what he means is that mathematics has been abstracted from sensible matter. And when he says the objects of mathematics rely on matter for their being, this simply means the objects are intelligible matter; which specifically is the quantity inhering in substance that may be considered without the sensible, despite being always together. This shows the force of the threefold division of the sciences since natural science relies on the sensible and metaphysics goes beyond the study of even intelligible matter.

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[32] Aquinas, Thomas. *The Division and Methods of the Sciences*. Pg. 41.

The reason that the study of motion does not belong in mathematics is expounded upon in the fifth objection to article three of question five. The objector claims that because in every motion a body will travel a quantity of space at a certain quantity of speed, the mathematician must study motion to study these quantities. The implication here is that the quantities of the space and speed must remain defined by the motion, not abstracted from it. This is noteworthy given the similarity of the objector's position to Descartes. Saint Thomas replies by saying:

> By its very nature motion is not in the category of quantity, but it partakes somewhat of the nature of quantity from another source, namely, according as the division of motion derives from either the division of space or the division of the thing subject to motion. So it does not belong to the mathematician to treat of motion, although mathematical principles can be applied to motion.[33]

In other words, since motion is only partly actual, it is not quantity, nor can quantity inhere in it; quantity must inhere in the space or the object moving. It is by these quantities *in* the space and the object that motion can begin to be quantified. Therefore, it is not the motion that determines the quantities, but the mathematical principles that determine the motion.

Further, motion is said to be the cause of the mathematicals insofar as they rely on matter for their being. For instance, one can really say a circle has been generated with the compass, since the quantity of the circle did not exist on the paper till drawn. However, one cannot say that this motion is the cause of the definition of the circle. For, the mathematicals are prior in definition to matter and motion. Thus, motion cannot simply be said to be the cause of the mathematicals.

[33] Aquinas, Thomas. *The Division and Methods of the Sciences*. Pg. 43.

From this, it is apparent that when Descartes declares that geometry is what "is precise and exact,"[34] he is not perfectly accurate. For though mathematics is precise and exact, it is only a precise science because it has been abstracted from matter and motion. Therefore, it follows that if a concept used in mathematics was found to have motion in its definition, although precise and exact, the concept would not be mathematical but belong to some other science.

IV. The Mathematical Intelligibility of the Function

Concluding from Saint Thomas, we now know that it was incorrect for Descartes to have established his geometry on the premise that motion may define the mathematicals. Motion simply is not the object of mathematics. Yet both Descartes and his alleged ancients maintained that Descartes' new curves are defined by motion and this has not been refuted. This raises the question: are these curves not mathematical? If so, the study of curved lines will not belong to mathematics but to natural science or even some intermediate science[35] between them.

To best answer this question, one should turn to the function. As shown above, the function arose directly from Descartes's new mathematical notation and now serves for the modern mathematician as a quasi-definition for curved lines. And it is not inconceivable for one to suggest that the function involves motion. Indeed, the idea of the function seems to have come directly from Descartes' imagined motion. This assertion is even validated by the definition of the Latin word *functio*, meaning a carrying out (of something). Thus, one is left

[34] Descartes, Rene. *The Geomerty of Rene Descartes.* Bk. II Pg. 44.

[35] Saint Thomas says, "the measurements of motions are studied in the intermediate science between mathematics and natural science: for instance, in the science of the moved sphere and in astronomy." This science Saint Thomas describes is modernly called physics. An extended discussion of the intermediate sciences, however, is beyond the scope of this thesis. (Ibid. Pg. 43.)

wondering if the function is mathematical. Gottlob Frege answers this dilemma in his essay *What is a Function* by showing that the true distinctive mark of the function is the law governing the relation of the variables.

Frege believes that the main difficulty with the idea of the function is that it is not clearly understood by the mathematicians. He says that "the name 'function' is given sometimes to what determines the mode of dependence, or perhaps to the mode of dependence itself, and sometimes to the dependent variable."[36] To state it differently, some think the relation between *x* and *y* is the function, and others believe the y or f(x) is the function. This is the heart of the problem: if the curved line is to be defined the function which is nothing other than the variable, then it will be mechanical.

One difficulty with defining the function as variable is that quantity, the subject of mathematics, does not vary. Frege himself states this, saying "what remains the same when a number varies? Nothing! Hence a number does not vary; for we have nothing of which we could predicate the variation."[37] When water becomes vaporized, one observes that something is underlying the change; if this were not true, one would not say that the water is now the vapor. However, this cannot be the case with quantity. For instance, π is no longer π when changed to 4. Yet even continuous quantities cannot be variable, since these quantities cannot be equal with their prior value or position if changed. This agrees with Saint Thomas since he would say that the mathematicals have been abstracted from matter and thus have nothing to underlie their change. Hence, the variable quantity mathematically cannot exist.

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[36] Frege, Gottlob. *What is a Function*. Pg. 230.
[37] Ibid. Pg. 231.

Thus, Frege argues that the function is unintelligible to mathematics if defined by the variable. The y= *f(x)* does not tell one anything substantive about the function itself. It is only when *y* is given a specific relation to *x* that the function becomes intelligible. For instance, $y=x^2$ or y=sin(x) both convey definite meaning in mathematics; y=f(x) does not. For this reason, Frege declares:

> Correlation, then, takes place according to a law, and different laws of this sort can be thought of. In that case, the expression 'y is a function of x' has no sense, unless it is completed by mentioning the law of correlation… And surely the law, which [defining the function as a variable] treats as not being given, is really the main thing. We notice that now variability has dropped entirely out of sight; instead, generality comes into view, for that is what the word 'law' indicates.[38]

This is because law is not variable, but motionless. Indeed, this is the same realization Descartes had when he realized that he could solve the locus problem by giving a definite relation; if this had been variable, he could not have given it as a solution. Further, functions with differing laws exhibit "qualitative differences."[39] This is nothing other than to say that functions such as x^2 and sin(x) will differ by their corresponding curves. And conversely, what really differentiates distinct curves is not the variable, but rather the law that governs the relation of the variables.

Frege additionally clarifies that "the sign for a function is 'unsaturated;' It needs to be completed with a numeral, which we call the argument-sign."[40] The reason that Frege calls the function's sign or the law here unsaturated is its ability to be completed by numerous values, thus, giving it a particular value or *saturating it*. Thus, law transcends any of these saturated answers, since it is what is common to all the answers. Notice also that Frege has called the independent variable the argument-sign; he does this because he sees that the function, when

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[38] Frege, Gottlob. *What is a Function*. Pg. 233.
[39] Ibid. Pg. 234.
[40] Ibid. Pg. 234.

considered in light of the variable, is a kind of argument. In other words, when given a value for the independent variable, the dependent variable will provide an answer.[41] Therefore, if the function is defined by the variable, then it will be nothing other than a sort of mechanism for providing answers.

Frege attempts to root out the confusion between the function as law and the function as variable by considering the benefits of a change in notation. For if the variable could be removed from the function so that the unsaturated-ness could be in full display, then the meaning of the function would be clarified. Frege suggests the best notation "for avoiding the confusion that arises from regarding the argument-sign as part of the functional sign"[42] is the replacement of the variable with empty brackets.[43] This is because this notation makes the unsaturated-ness of the function sign most evident. Another possibility he suggests is the use of the symbol 'ξ' instead of the variable when wishing to express the generality. Yet all "'ξ' does here is to show the places where [particular value] has to be inserted."[44] Or in other words, there would be no real essential difference between 'ξ' and the variable or argument-sign; 'ξ' would still be just a place for the insertion of a particular value for the argument. Yet, one could further make the same argument for the empty brackets, and hence the function would remain untreated.

[41] In calculus, the variable occupies the role of argument sign as well. An indication of this is how taking the limit of a constant does nothing. Thus, the variable enables the argument, which is implicit in the taking of the limit, to arrive at an answer.
[42] Frege, Gottlob. *What is a Function*. Pg. 235.
[43] "e.g. 'sin( )' or '$( )^2$ +3( ).'" (Ibid. Pg. 235.)
[44] Ibid. Pg. 235.

Perhaps the real source of this difficulty, however, is not what specific notation is used but rather the use of symbols itself.[45] The application of signs, such as *x* and *y*, makes it incredibly convenient to mechanically plot a curve on a graph. Yet, this use of the function as an argument obscures one's understanding of the unmoving nature of the law that governs it. It was conceivably this very dilemma about the function that led Descartes to believe that curves are defined by motion. This is the real wisdom of Apollonius' use of words like *abscissa* and *ordinate*. Words convey a definite meaning about a thing's essence, which symbols are incapable of doing. The author Charles De Koninck speaks at length about this in *the Hollow Universe*, stating that "the ancient distinction between what is *per se*- either in being, in unity, occurrence, &c.-and the abysmal *per accidens*… is largely ignored nowadays…. Failure to make [this distinction] must lead to the identification of name and symbol; of logic and mathematics with the *mechanics of computation*."[46]The difficulty with the use of signs is that all that they represent does not need to be *per se* one. A symbol can represent an accidental heap.

Hence, if the curved line were to be defined by the variable or a function used as an argument, then the curve will be reduced to nothing but a sum of its points. For by leaving the points only accidentally a whole by the symbol f(x), the unity of the line is destroyed. Consequently, the loss of unity in the curved line causes it to lose its intelligibility. For Aristotle says that a line "to whose divisibility there is no stop… we cannot think [of] if we do not make a stop."[47] In other words, the line as an infinity of points is unknowable. Additionally, the curved

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[45] I diverge here from Frege's argument. It is unclear whether he believes this difficulty with the function arises from the use of symbols. And although Frege clearly believes that law is what the function is, he makes no mention to whether he believes curved lines to be defined by this law.
[46] De Koninck, Charles. *The Hollow Universe*. Pg. 11. Footnote 1. Italics added
[47] Aristotle. *Metaphysics*. Bk. II. Line 994b22-24.

line is only really alluded to by the potency of the function, if thought of as an argument, to be answered by an infinity of particulars. What is more, it is impossible to calculate every possible answer. Plot as many points using the function mechanically as one wishes, and one has still failed to reach the actuality of the curved line.

Aristotle solves this difficulty above by stating "the whole line also must be apprehended by something in us that does not move from part to part."[48] Put differently, the human mind when it abstracts the mathematicals understands them as one by the unity of their definition. This is the reason that when one first encounters a particular function, one does not need to find every single point to imagine the curved line. If one can discover key points on the curve,[49] then the human mind will be capable of filling in the rest of the line.

Consequently, what has been lost in the science of mathematics when the curved line is thought of as defined by the function used as an argument or any mechanism is that element of the science of mathematics that makes it a truly human science; namely, the understanding of the being of the thing rather than mindlessly computing.[50] Though the function mechanically can never actually plot an infinite number of points; the human mind, peering beneath these finite points, can behold the pattern or the law or *the form*. This is *the true object of Mathematics*. The function as an argument cannot be an object of mathematics since it is always in potency to describing the form.

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[48] Aristotle. *Metaphysics*. Bk. II. Line 994b25-27.

[49] For instance, a rough drawing of the curve can be made if one knows the x or y intercept, maxima, minima, points of inflexion, etc...

[50] De Koninck emphasizes this point, saying "computation… does not require understanding of what is being computed. Even when performed by a man, a computation can be purely mechanical and, in fact, ought to be. But I am sure you will agree that a curious compliment is being paid to the human mind by the assertion that [computation's] supreme degree of precision is achieved when nothing is either apprehended or stated, and when reflection is more of a hinderance than a help." (De Koninck, Charles. *The Hollow Universe*. Pg. 28.)

Therefore, what Descartes has failed to see is that a function, if defined by the variable, or any mechanical device can never be an adequate definition or species making difference for a curved line. Although the function used as an argument is excellent for plotting and discovering new curved lines, a definition for these lines will always be only hinted at in this argument, namely by the unsaturated-ness of the law governing this argument. And because law is unmoving, it will be appropriate mathematically to define curved lines by it.

Additionally, it has now been shown that motion cannot be a basis for any division of curves since motion does not define them. Hence, it was wrong for Descartes to bar curves plotted by transcendental functions from his geometry based on what motions would be required to describe them. Although transcendental curved lines, insofar as the functions describing them contain an infinity of algebraic operations, will be to a degree unknowable, they still have a definite law of correspondence which can be studied by mathematics. Thus, Descartes was right to allow his new curves into geometry since they are truly mathematical; but he was wrong to have invited motion in with them.

V. Motion's Role in our Mathematical Knowledge

This conclusion leads one to an important question: how did Descartes and seemingly all modern mathematicians fall into the error of not separating motion from mathematics? Why is this such a common mistake? The answer lies in what man's mode of knowing is best able to comprehend.

Saint Thomas in a reply to an objection gives further reasoning for the ordering of his division of the sciences. The objector claims Saint Thomas' ordering is wrong given that mathematics should go before natural science. The objector states this because children can easily learn mathematics, yet natural science requires time and experience. Saint Thomas replies by saying, “Although we should learn natural science after mathematics because the general proof of natural science requires experience and time, still, since natural things fall under the senses, they are by nature better known than the mathematical entities abstracted from matter.”[51] In other words, motion and matter is more known to man because it appeals directly to ones senses. Hence, man’s mode of knowing is better able to understand motion than the mathematicals, despite the mathematicals being of a more knowable nature. This can be easily seen by how even the most illiterate people over time gain basic knowledge in the motion of bodies, and yet will have no knowledge of mathematics unless learned.

One consequence of this is that motion at times may be an excellent method for understanding mathematical truths. For instance, the easiest way to see the definition of the circle is by observing the motion of a compass. For, if the only circles that a man observed were the disks of the sun or moon (or some other circle that has not been generated), it would not be as clear why the center and radius should be included in the circle’s definition. On the other hand, if one saw the motion of a compass, one would immediately understand that the fixed arm and the fixed distance between both arms is the reason the circle is described. Thus, this man has come to comprehend the circle first by motion.

One reason for why the circle can be better understood using motion is that the circle can be described as a locus. Perhaps this is the cause of why Euclid chose to define the sphere

[51] Aquinas, Thomas. *The Division and Methods of the Sciences*. Pg. 24.

by the revolution of a semi-circle[52] or Apollonius chose to define the conic by the revolving straight line through a point and circle.[53] Both can be more mathematically expressed, namely by understanding them as loci. Yet the beauty of their definitions is that by using motion, the infinity of solutions to the locus can be grasped as a whole.

Motion is used similarly in calculus. Although the limit, at least in itself, is devoid of motion, motion is used as an aid for the mind to grasp it. For Aristotle says that one meaning for the word limit is "the first point beyond which it is not possible to find any part, and the first point within which every part is." Because there can be no last part prior to the limit, one may divide this quantity prior infinitely. Thus, calculus only employs motion to help one's mind extend over the infinite preceding the limit. And certainly, the tangents to curves and the areas underneath them are unchanging values and thus are objects of mathematics. Calculus only uses motion to understand how to derive them.

For these reasons, all curved lines can be better understood by way of motion. This is also the reason that using the function has become the predominant method for understanding curves. And in fact, this is what made Newton's and Descartes' new methods so effective at solving mathematical problems. Consequently, mechanics may at times serve geometry, though in no way is geometry simply a part of mechanics as asserted by Newton. Therefore, the basis

[52] "When, the diameter of a semicircle remaining fixed, the semicircle is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a *sphere*." (Euclid. *Euclid's Elements*. Bk. XI. Definition 14)

[53] "If from a point a straight line is joined to the circumference of a circle which is not in the same plane with the point, and the line is produced in both directions, and if, with the point remaining fixed, the straight line being rotated about the circumference of the circle returns to the same place from which it began, then the generated surface composed of two surfaces lying vertically opposite one another, each of which increase indefinitely as the generating straight line is produced indefinitely, I call a conic surface…" (Apollonius. *Conics: Book I-IV*. Pg. 3. Definition 1.)

of the error that motion must define the curve is rooted in failing to understand that these motions are just an analogy that allows man's mind to understand complex truths. In no way do these motions define the mathematicals, but they are an excellent way to grasp their definitions.

CONCLUSION

To conclude, motion cannot serve as an adequate definition of the curved line. This is supported by Saint Thomas' account of mathematics because according to him mathematics studies quantity rather than sensible matter and the motions belonging to it. Further, it is law that defines the curve, not any mechanism. And finally, the source of Descartes' error is that he failed to see that motion is only a device the mathematician may utilize to exemplify the definition of the curve.

Thus, the science of mathematics has not really undergone any essential changes. Although Mathematicians have refined their tools and mechanisms for discovery, their object remains the same. Indeed, Descartes' new mechanisms are fruitful for a good reason. For, motion has a powerful role to play in the science of mathematics. If one were to speculate on the future of mathematics, one may guess motion will play a definite role in the science's future developments.

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